

PHILOSOPHY OF FINANCE

# Definitive Smile Model: Part I

Why should we write about smile models? This is the question behind the question. For if the definitive smile model is not yet in sight, perhaps a definitive smile story is possible.



What is there more to say on the subject of smiles, and what is there to expect from reflection on the smile problem today? Could the answer be the further elaboration of the existing models? That is, could the future of our story be purely technological and one of taking up the technical complications one after the other, trying out jump-diffusion after the diffusion or stochastic volatility after local, deterministic volatility? Should one become a specialist in Laplace and Fourier transforms, and rank the models by classes of integrability, carefully selecting the functional form that promises the most exciting analytical gymnastics? And shouldn't then every quantitative analyst start worrying about the best way to promote his model, and how to best argue that his model must be the right one? Jump-diffusion may be better than diffusion because of the existence of large and rare moves in the underlying. Moreover, "the ability of infinite-activity jump processes to

capture both frequent small moves and rare large moves" may give us a further reason, as argued by Carr et al. (2002), to discard the diffusion component altogether in the light of statistical evidence for the fine structure of asset returns. Or perhaps the quant should worry about explaining market option prices as instantly observed rather than analyzing the underlying time series, and feel confident that his smile model is the right one when it is able to match the prices of, say, the barrier options, on top of the vanillas. This is the point of Lipton (2002b), and his defense of his 'universal volatility model' which mixes jump-diffusion and stochastic volatility.

### Lipton's progress

"Why should we write about smiles anymore?" The answer may be that the only thing worth writing today is a review of existing smile models and their classification, *à bestiaire*, like the French say. This is what Lipton has attempted (2002a). A roadmap may indeed become desirable when the territory keeps expanding and the beasts look

stranger and stranger, if only because it has the virtue of listing the known obstacles and the dark alleys. You read here and there that closed-form solutions cannot be had when there is correlation between the underlying and its volatility, or that calibration becomes a formidable task when the underlying is jumping and volatility is stochastic. A roadmap, however, is only as good as the vehicle that it is intended for, and it is clear that Lipton's intended vehicle is the closed-form, or semi closed-form solution, when it can be had. On the other hand, there is something disheartening about the

very idea of a 'complete guide,' and that is that such a guide is only as good as its vintage. Apart from proposing a smile model for every taste and culture (jump-diffusion, stochastic volatility, local volatility), and updating us on the last fashionable trend, what is to be gained from such a listing over and above its comprehensiveness and good taste? What is the real advance? And when the 'universal volatility model,' that Lipton offers for the finale of his catalog, is itself interpreted back into the series as the latest model produced, or in other words, the last model of the list which naturally beats all the others in terms of complexity and number of parameters, might we not fear that the truly different argument that Lipton brings up at some point, namely the capacity of this model to match the market price of barrier options, may look very remote? If matching the barrier option prices is such a definitive argument, then why bother with the history and lineage of smile models anymore? Lipton's dramatic build-up makes it all sound as if smile modeling finally reached an age when jumps can be safely

combined with stochastic volatility and the appropriate Fourier transform successfully obtained, and as if – surprise! – the market concurred, in celebration of that age and in acknowledgement of that maturity, with the gift of his agreement on the barrier option prices. Are we to believe that empirical agreement with the barrier option prices was just waiting for this last advance in smile theory and smile model design, and for an advance with precisely that parametric form? Or was the ‘universal volatility model’ somehow encoded in the market? What other reasons are we offered for this agreement apart from pure luck, or just the supernatural argument that the ‘universal volatility model’ is next on our list and has got to address, for that sole reason, the next unsolved problem which is the matching of barrier option prices? Instead of showing us what it takes for a smile model to match the barrier option prices over and above its matching the vanillas, and why the ‘universal volatility model’

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has it – for that would provide a real next stage for our thinking about smiles – Lipton lapses into metaphysics and retains as the only benefit from agreement with barrier option prices the fact that his model must somehow be distinguished on that list and be true; period. To put it differently: What if barrier option prices contained additional information that has to be calibrated in the model independently of the vanillas? When Lipton’s point is precisely that two smile models can agree on the vanillas and disagree on the barriers, might we not fear that the market dynamics may evolve the next day and imply a different price structure for the barriers, given a certain price structure of the vanillas? Might the ‘universal volatility model’ not fall into disgrace itself despite its superlative name, and the market favors shift to a more

encompassing model still, or perhaps revert to an older and simpler model? Again, what is missing here is a theory of that extra step, or new frontier in smile intelligence, which the barrier options represent, and empirical evidence is just not good enough an argument.

### A meta-model

It may sound as if I am hinting at some kind of superior model, or meta-model, which could see what is happening when the ‘universal volatility model’ manages to match the barriers and the Heston model, or the local volatility model, do not. It would be a meta-model both in the sense that it embeds the models of lower rank as specific instances and that it provides a critique of those models. But then the ‘universal volatility model’ was supposed to be just that! As a matter of fact, ‘universal volatility’ is not just a model, but a whole family of jump-diffusion models combined with stochastic volatility and it can reproduce, at

one extreme, a local volatility model or a pure diffusion with variable diffusion coefficient, and at the other, pure stochastic volatility or a Heston-like model. It can even assume a pure jump process. So while Lipton has proposed the all-encompassing, overarching model we are looking for, he has not provided the critique. And the reason is that he paused at the meta-level only to rush down into the one instance of his meta-model which afforded an analytical solution, yet differed enough from Heston or local volatility to deserve the name of ‘universal volatility model.’ More importantly, Lipton has not taken the extra step of calibrating his model to the barrier options a priori. The vanilla implied volatility surface is all we have to go along with in order to establish the parameters of the model,

and agreement with the barriers is then checked a posteriori. We are left with the flat conclusion that his vanilla-calibrated ‘universal volatility model’ predicts the right price for the barrier options, for example the double-no-touch, for no other reason than that it hits the right balance between the local volatility model which underestimates it and the Heston model which overestimates it.

### Beginning of the smile problem

So at best, Lipton’s ‘universal volatility model’ looks like an adjustment or a refinement of pre-existing models. “The market is subtler than you think, so the story goes. It doesn’t exactly behave like any of the standard smile models you’ve been using, local volatility, stochastic volatility, pure jump, but somewhat in the middle. And what else did you expect? The road to barrier options has been concealed from the known tracks, but it definitely exists on our roadmap. This is precisely the road that you can see now opening up in the middle. It may be a little harder to journey because of the additional parameters and the tougher Fourier transform, but it is there alright.” The reason I dispute this statement is that the ‘universal volatility model’ is not in the middle, but is supposed to be above. It shouldn’t really belong on the roadmap, but in the bureau revising the roadmap. And the barrier option pricing problem is supposed to be the key to our real thinking about smiles, and not just fall as an additional item on the list of things that one model can do and the other cannot. As long as the smile problem was one of accounting for the implied volatility smile of the vanillas, alternative explanations could compete on the same level and their relative advantages be compared. One explanation, for instance, proposed that the coefficient of the Brownian diffusion was not constant in the plane but varied according to Dupire’s formula. Another claimed that the diffusion process was overlaid by Poisson jumps, whose size and intensity we would have to determine by calibration. Yet another assumed that volatility was stochastic itself and correlated with the underlying. Or indeed an explanation mixing all three kinds of process, diffusion, jumps, and stochastic volatility, could be considered in turn. Any of these explanations was

as good as another as long as the challenge was to describe a certain way that reality should be, for a vanilla smile to be the consequence. You may have had issues like over-fitting or under-fitting and questions about the right number of degrees of freedom and whether or not you should allow for term-structure of the parameters, but these were technical issues.

However, the smile problem enters a new phase – or rather, it rises to a new level – when it becomes one of trusting the proposed model for the hedging strategy one should follow. In fact, the vanilla option deltas produced by the competing models differ largely from one model to another. For instance, the local volatility model predicts that the option price will evolve with the underlying in such a way that the smile moves in the opposite direction to the underlying movement. See Hagan (2002) for the analysis and criticism of that phenomenon. By contrast, a stochastic volatility model, like Heston or SABR, predicts that the smile evolves in the same direction as the underlying, or in other words, that implied volatility is a function of the option moneyness. From descriptive metaphysics the problem has now moved to speculative metaphysics. The question is no longer to explain the present smile, but to predict its evolution. In fact, the smile problem, as I like to call it, really begins here. Indeed, any of the static descriptive explanations of the vanilla smile is as good as another, and for that matter, no better than straightforward spline interpolation! No one would have a problem with the smile, and no one would need a smile model, if the problem was just the pricing of vanilla options under implied volatility smiles. Similarly, the smile problem really begins with the question of pricing the barrier options. Since there is no way we could interpolate a Black-Scholes implied volatility number for the barrier option from the vanilla implied volatility surface – should we interpolate at the strike of the barrier option or at its barrier? – we definitely need a smile model to form its price. And surely enough, the vanilla-calibrated competing smile models yield different barrier option prices, just as they yield different vanilla option deltas. Speculative metaphysics back again.

The term ‘metaphysics,’ however, seems to suggest that the truth must be lying somewhere

behind the phenomenon, only we have no other way to get hold of it at present but to speculate about it. And now Lipton’s article on ‘universal volatility’ and Hagan’s article on SABR appear as ways of re-embedding speculative metaphysics into descriptive metaphysics, by enlarging the view. Both authors argue that their model describes reality accurately, only they draw a more comprehensive picture of reality. Their picture now includes, beside the vanilla smile, the observed barrier option prices in Lipton’s case, and the observed vanilla option deltas, in Hagan’s. Both authors seem to ignore the possibility that the barrier pricing problem, or the vanilla delta problem, may be adding a new dimension to the smile problem rather than a new side to reality, and that both the barrier price structure and the vanilla delta structure may change, for a fixed vanilla smile. What would Lipton do if empirical barrier option prices moved closer to the pattern predicted by a local volatility model and away from his ‘universal volatility model’? And what would Hagan do if empirical vanilla option deltas started reflecting a sticky-strike situation rather than sticky-delta? Would they discard their models? As a matter of fact, different delta behaviors and different barrier price structures have been empirically observed at different times and at different places. See Derman’s paper on volatility regimes. In the end, Lipton and Hagan may be just reflecting a reality specific to their particular market, foreign exchange options in Lipton’s case, and interest rate options in Hagan’s. (Even worse, they may be reflecting a self-fulfilling prophecy). In other words, it may very well be that the vanilla option deltas have to be calibrated into the model independently, the same way the barrier option prices should be. Indeed, we show in another paper that the two problems are intimately linked, and that they hinge on the dynamics of the smile.

“What is there more to say about smiles?” And the answer should be: Everything! Any smile model leaving untouched the question of the hedging strategy of the vanillas, or the question of the pricing rationale of the barrier options, has not even begun to address the smile problem. And it will not do to argue that the vanilla hedges have consistently been observed to be such and such in my market, or that the barrier option prices hap-

pen to be such and such. The fallacy which consists in arguing for the validity of a given smile model (‘universal volatility,’ SABR) on the grounds of the empirical confirmation of the option delta or the barrier price it produces, is worse than leaving these problems untouched. For it suggests that all there is to expect from the delta or the barrier price is a distinction and a confirmation in retrospect, and that agreement with the market delta or the market barrier price is the last word in the smile model contest. It suggests that the problem is over, when we claim that it has only begun and that the delta and the barrier are the first things we should really be writing about.

### **A departure from Black-Scholes**

We may essentially define smiles as a radical departure from Black-Scholes. And we do not mean it in the sense that the observed vanilla prices differ from the Black-Scholes uniform implied volatility. For all we know, the Black-Scholes formula may have never existed. It may have been altogether unimaginable that re-hedging could take place continuously or that transactions could be costless. And Black and Scholes, for that matter, may have had to come up with a more complex formula, which implied itself a ‘volatility smile’ relative to the usual formula. What we mean when we say that smiles are a radical departure from Black-Scholes, is that smiles really begin when we are no longer able to apply what is really important in Black-Scholes. And what is really important in Black-Scholes is not the formula or the usual simplifying assumptions (continuous, frictionless trading) but the following two things: the dynamic hedging idea and the idea of translating the option price into an implied volatility number. These are the true inventions which have revolutionized our way of dealing with options.

Now translating the vanilla option price into an implied volatility number is still possible under smiles: interpolation does that nicely. Therefore the smile problem doesn’t begin here. The smile problem begins as soon as we depart from Black-Scholes and no longer have a fix on either the hedge or the representative volatility number. It begins with the problem of the vanilla option delta and the problem of the barrier option price representation. This is the reason why any smile

model that manages to match the market prices of the vanilla options, but offers no guarantee that it will match their market deltas, or that it will match the market prices of barrier options, really ends before the beginning of the smile problem. Lipton and Hagan offer no such guarantee. They are just lucky enough that their model agrees with their market reality. The only way to offer the guarantee is to build it into the model. This is a call to a voluntarist and active attitude. And now we can understand why Lipton and Hagan, who had no means of controlling the barrier option price structure, or the vanilla option delta structure, beyond the matching of the vanilla option prices, could offer no other guarantee than just the passive belief in the existence of a truth out there and the correspondence of their models with that truth.

### Thinking after Black-Scholes

It will be my contention that Lipton and Hagan are the last representatives of a philosophical tradition that misinterpreted the meaning of the Black-Scholes model and the significance of its teaching. Philosophy and interpretation wouldn't worry us much if they had no effect on the science and remained confined in the preserve of reflection and meditation. It doesn't really matter to the Black-Scholes model how we interpret it or philosophize about it. The philosophy of Black-Scholes (and more generally, the philosophy of derivative pricing) will be shown to matter, however, to the science and practice that followed Black-Scholes, namely the smiles. The smile problem, as we face it today and insofar as it begins today, is essentially a philosophical problem. Or so I will argue. To really think about smiles, one has first to learn to think about Black-Scholes, and only then will one know how to think after Black-Scholes. Since smiles are the radical departure from Black-Scholes, anyone misinterpreting Black-Scholes will misconstrue the way of departing from it, and therefore will misunderstand smiles.

'Departure from Black-Scholes' and 'thinking after Black-Scholes' have to be understood in the two senses of the terms. Smiles depart from Black-Scholes in the sense that they radically differ from it and that they take in, basically, anything that constitutes a breach of the Black-Scholes para-

digm. (And it is a big world out there! Jumps can induce volatility smiles, but so can stochastic volatility, and default risk, and firm leverage, and discrete hedging, and transactions costs. Any realistic derivative pricing model is a smile model, really). And smiles depart from Black-Scholes in the sense that they issue from it and that they are its generalization. Or rather, they will strike us as the true generalization of Black-Scholes, once we identify the strands in Black-Scholes that should really be generalized. Likewise, thinking about smiles is thinking after Black-Scholes: thinking what is next and taking up where Black-Scholes has left off. And it is thinking after Black-Scholes: thinking in the style of Black-Scholes and following its teaching.

Now the reason why the tradition that followed Black-Scholes has misinterpreted it and missed the thrust of the whole new science that was being born, is that it thought of the Black-Scholes model as the description of some physical reality. It thought Black and Scholes were literally after the lognormal distribution of asset returns and presumed that the Black-Scholes model was false when it was faced with the first deviation from the predicted option prices, i.e. smiles. Yet this tradition had nothing to say about the widespread continued use of the Black-Scholes pricing formula in spite of the obvious inaccuracy of the underlying theoretical model, or about the apparent ease with which traders just went ahead and plugged in a different implied volatility number for every different option they wished to price. This phenomenon is sometimes referred to as "the robustness, or the resilience, of the Black-Scholes model." The traditional criticism explained it away as being just a consequence of the simplicity and intuitive appeal of the Black-Scholes model. And while it set out to find the theoretical substitute of Black-Scholes, it argued that people using Black-Scholes were doing something they shouldn't really do. The situation was one of essential tension between the longevity and increasing popularity of the Black-Scholes model (still the textbook model, still the option pricing benchmark) and the increasingly smaller odds that the 'true' model may finally be found. For once correspondence to truth had become a requirement and once the alternative to a false Black-Scholes had

been philosophically reduced to the quest for the true smile model and nothing but the true smile model, this quest could not just stop at the first step and simply match the vanilla smile. The true model had to tell all the truth: it had to match the barrier options, it had to produce the right hedges (witness the arguments from Lipton and Hagan), and last but not least, it had to appeal to practitioners, not just academics, and satisfy them that it was every bit as robust and functional as Black-Scholes.

Never before in the sciences had we witnessed such a big gap and such a great conflict between the endeavor of the theorist looking for the true model and the behavior of the practitioner using the model. While the continued 'falsification' (to use a Popperian term) of every successive model had done nothing but excite the theorist and exacerbate his belief that the truth must be lying ahead – forever lying ahead, never in the present model, always in the next – and while it had done nothing but precipitate an escalation of arguments from his part instead of making him consider a radical alternative, the practitioner had no such exacting concerns and enjoyed a much greater freedom of movement, literally making the truth rather than finding it, and making the market in the vanillas and the exotics. Not mentioning that the exotic structures themselves were being made up every day and that they created new markets every day. So are we to believe that truth is just sitting there, waiting for the true model to find it, and that this moment of truth will then at once embrace all the exotic structures that have come about or will have to come about? Or might the theorist argue that truth is itself a relative and forever shifting notion and that he doesn't mind reiterating the whole nested sequence of models every time a new class of exotic structures is introduced, no matter whether the new sequence and the new 'history of science' contradicted the previous ones? And how would we account for the transition regimes, where truth is not yet itself an established notion and the only truth-maker is everybody's guess about what to count as an arbitrage?

● *Continued in next issue*