THE MEDIUM OF CONTINGENCY

Dan Tudball discusses the consequences of inverting our thinking about the market with Elie Ayache

In *The Medium of Contingency*, Elie Ayache continues the project he began in *The Blank Swan*, the book in which he argued that our current approach to derivatives pricing rests on an application of probability that is self-defeating. That isn't to say that probability isn't relevant at all to the process, just that its implementation is transgressive, according to Ayache's inverted methodology.

Perhaps the easiest way to appreciate this particular writer's work is to understand that inconsistencies don't sit well with him – and that no matter how easily it seems that the inconsistency in question might be glossed over, he will examine it, pick over it, and deconstruct it to the minutest detail. This loathing of inconsistency is, in Ayache, quite pathological but in a most civilized way, and why not? He loves the market but he will not accept explanations of it which will 'just have to do,' approaches to it that require a suspension of disbelief (or belief). This is because the man likes to think, too, and some might argue that he thinks to a fault – because the places his thinking takes us can be pretty alien.

In actual fact, Ayache offers a number of 'ways in' to his world, and they all share the common feature of being some form of paradox. Maybe Ayache is himself a paradox, in that he is a derivatives market technician who writes philosophy, or vice versa; whichever it is, he is certainly opening an invitation to both sets to examine what the other hath wrought, as it were. His door in for the philosophers of probability is the suggestion that the market itself is a massive challenge to traditional thinking about probability. What about market practitioners, what's the way in, which will allow them to feel at home in a discussion that owes its logical force as much to the philosophy of semantics as it does to familiarity with Black–Scholes?

ELIE AYACHE

How about this seeming truism, lifted from *The Medium of Contingency*, which serves to express the tension that market practitioners skim over every moment of every working day:

> "*You maintain that hardly anybody assumes a 'random generator'. Well, do they? Apparently you haven't been talking to quants, or attending quant conferences, or reading any theoretical work in finance (not to mention econometrics) over the last decades.*
>
> '*Oh, but traders and market practitioners "without the elaborate vocabulary" (Brooklyn boys, as Taleb would call them) do not assume random generators or fictions of that ilk! They just trade the stuff; they don't model it or theorize about it.*'
>
> '*Fine; but, then, what do you make of the builders, like myself, of a technology of derivative pricing? Where do you rank technology? On the side of theory or the side of practice?*'
>
> *Quants can forever deny recalibration and living market-makers can forever deny models. But the technology sits right in the middle, or right in the knot, and it dictates that recalibration should become a technological process and should be* re-embedded in the technology."

Where *The Blank Swan* was an introduction to Ayache's problem with probability, and an exploration of how a reconstruction of the meaning of the market and prices might get us out of a logical bind, *The Medium of Contingency* is the user manual for a technology – that technology is the market for contingent claims. Remember that a contingent claim is Ayache's preferred name for what we call derivatives. Why not derivatives? Because that name connotes that the asset is a function of something else and not an independent tradable entity. This is all-important; names are important – they lead to associated ways of thinking. As long as you think of a derivative as a function of something else – like an underlying – you're not really saying what it really is, which is a tradable asset with its own market, with its own price, a price that actually has very little to do with the underlying – let alone a price that is relegated to being a function of it. You know that paradox, right, that whole Black–Scholes thing? Better to call it a contingent claim, a more accurate description of an asset that pays off a certain amount should a certain contingency arise. Finding a way in yet?

So, Ayache is presenting a technology. And why is the Market for Contingent Claims a technology? Because the definition of a technology is the combination of theory with material procedure. In the case of the market for contingent claims, this is the combination of derivative valuation theory with the recalibration of the theoretical valuation model. Ayache, with his hat on as a builder of derivative pricing technology, presents the sharp end of the problem for him; the aim in providing a derivative pricing technology is to solve the recalibration problem. You know – the thing that quants deny, lying on the other side of the model that can't ever quite get out of its own feedback loop?

Ayache has found something, and I leave it to him to describe this historical discovery – an artifact from the days when options pricing remade itself in the shape of Black–Scholes; an artifact that, according to Ayache, will allow for a consistent story to be told about the marketplace and might actually allow quants and market makers (and philosophers of probability too) to finally hug it out.

After *The Blank Swan*

Dan Tudball: *What was it that occurred after the writing of* The Blank Swan *and the release of that book that brought you to this juncture, and what you're describing as your work being very much focused on semantics now; let's talk about that progression.*

Elie Ayache: First of all, semantics is a very important word for me, and a key word in my whole thinking. It goes back to 15–20 years ago, when I was engaged in the philosophical research of the philosophy of science generally, and the formalism of physical theory and how the formalism of a theory is interpreted. So, you have here the first opposition between form and matter, between the formalism and a model that the physical world or the material world is going to provide of that formalism and theory.

That's why, in the book, you find that I use another keyword – "matter" – and as a matter of fact the whole book is really aimed at arguing now that the price and the market are matter as opposed to pure speculation or pure probability.

That's how I relate to semantics – semantics in the sense of interpretation of theory; of really trying to understand what a formalism is trying to say – not in and by itself because in and by itself it's just symbolism and it's just syntax, so, obviously, we have to add matter to it in order to interpret it. However, it is true that it is not that you are interpreting in a kind of metaphorical way, we have to be using it, and the formalism is going to be itself indispensable in how we deduce matter from it. That's a general observation.

Dan Tudball: *At the core of the new book is the formalism of Black–Scholes.*

Elie Ayache: Yes. Another thing that has been on my mind, even before I started writing *The Blank Swan*, even as I started writing the first articles in *Wilmott*, back in the early 2000s; I was actually trying to understand implied volatility and trying to understand the formalism of Black–Scholes itself.

There is actually something that to this day puzzles me a lot in Black–Scholes, and I think that in the second book, *The Medium of Contingency*, I have progressed a little more than in *The Blank Swan* in ways of exposing at least the puzzle or the problem. That thing is, simply, what do we really mean by dynamic hedging? In which kind of time or register of time does the dynamic hedging occur? How is the time (time in the sense, really, of the dimension of time, of chronology) is the dimension of time in which the dynamic hedging occurs, not ultimately incompatible with the dimension of time in which the market of derivatives occurs?

Dan Tudball: *Do you manage to solve that paradox?*

Elie Ayache: That's one of my preoccupations today, and I probably have succeeded in framing the puzzle in the last part of the book, but I don't think yet the conclusion, so I'm still working on it right now. As you can imagine, as you can feel, it's pretty much related to the formalism because, obviously, the formalism is a formalism of volatility and stochastic processes which take place, at least formally, in time and on the other hand are used by traders who live in a trading pit because you are supposed to be dynamically hedging therefore you are supposed to be accomplishing the trading decisions of dynamically hedging.

I have always thought that being immersed in the trading pit is really living in a different category of time and dimension of time altogether than the one that gets formalized in mathematics. Of course, this is obvious to anyone; anyone will tell you that

Black–Scholes is only a model, but I don't think so.

I think it's more than a model because it is helping us to create a market of contingent claims. For instance, one of my claims in this book is that it is truly through the formalism of Black–Scholes, once you have interpreted it in the right way, according to me, that the market of contingent claims gets properly created and generated, even though it hasn't happened this way in reality.

Of course, in reality, you already have people trading options and then Black–Scholes theory happened in 1973 and you have a lot of books on the sociology of finance or history of finance that tell you how the discovery of the formula has changed the world of the traders and how the world of traders has more or less adjusted itself in order to make the formula possible; I'm relating, for instance, to the book by Donald MacKenzie, *An Engine, Not a Camera*, so that might be true in reality.

However, in my kind of semantic reconstruction or genetic reconstruction or genesis of the market – in other words, in the story that I'm trying to tell right about how we should understand the way that the matter that the contingent claims market is made of is deduced from the formalism – in this kind of reconstruction or rereading, this is the way that I see it now, really truly, the market of contingent claims, which in my sense generalizes to re-embrace the whole market, really gets deduced from a particular way of reading the formalism of Black–Scholes.

Dan Tudball: *Once again, you are asking us to take a very meta-view on everything. Is there some sort of disconnect between our current interpretation of formal models for derivatives and the matter toward which those formal representations are directed?*

Elie Ayache: You see why I'm playing now with concepts which may sound strange at the beginning, meaning that I'm trying to deduce something material and something that exists which is the market as we know it from a formalism, but I now believe that this is the right way to be understanding the formalism of Black–Scholes, and this is, by the way, why it differs a lot from physics and physical theory because no one would want to claim that you deduce the physical world itself from the formalism of the theories.

So, to go back to matter as opposed to form, first of all there is also a kind of dialectics that I introduce in the book that I think you have also picked up on in your notes between matter and the void; and also, at some point, I speak of matter being introduced because of its sharpness, because it's so sharp that this sharpness is equivalent to hardness and then to matter, in a way.

There is, first of all, a void – what I call the void of possibilities – from which I want to deduce matter, and this void is something which I have written about in *The Blank Swan*. When you categorize the world in terms of states of the world and in terms of the diagram of possibilities or in terms of a decision tree, like we all do when we are pricing stuff and simply sampling in our algorithms either the underlying or its volatility in case the volatility is stochastic or whatnot, every time you are modeling the unpredictable with some kind of probabilistic modeling, of course, you would be missing out a major event, which is, that something is going to become wrong with the whole thing, so the event – the true event, in the sense of the Black Swan, if you will, that Nassim Taleb speaks about – always comes from outside the model that you had, from outside the total of possibilities that you had, and, by definition, any kind of total of possibilities that you might have imagined will, by definition as if it were the opposite side of it, have something that can ruin it.

If you have an algorithm that is doing algorithmic trading and that is mapping out all the possibilities that the market is offering to it and assigning probabilities for those possibilities and, therefore, executing trades based on those possibilities and probabilities (even though it may be the best self-adaptive or self-learning algorithm that you have), the one possibility that, by definition, will be the other side of the algorithm is that the algorithm itself fails; that there is no way that a tree of possibilities, as complex as it may be, there's no way that it can have as one of its branches the branch that connects to the whole abolition of the tree, if you will. But you can, of course, make it simple and simply define the event as that, by definition, which emerges from the void of possibilities, meaning what is outside of the total of possibilities that you have up to that moment managed to model.

So, we can adopt that as a definition of the event. So, here again, and this is a project that I had already started in *The Blank Swan*, all I'm trying to do is to try, instead of always defining the event negatively as something that comes from the void or something that is always equivalent to the failure of the model of probabilities or the decision tree that you had, to turn this negative statement into a positive one. So, to turn the void into matter, and to say precisely, now, because of the fact that the event emerges from the void, and provided I can find the positive medium that can connect with that, and therefore is a medium of contingency that is, by definition, always external to a decision tree, and provided that I can find this positive medium, therefore construct a material bridge into that void, to say that maybe that would be the actual medium into the event, and this medium, I claim to be the market of contingent claims, for instance.

[B]eing immersed in the trading pit is really living in a different category of time and dimension of time altogether than the one that gets formalized in mathematics

Dan Tudball: *How does this bring us toward properly defining inconsistencies in our interpretation of Black–Scholes?*

Elie Ayache: I claim that, in the market of contingent claims, especially if you think of it through the technology that we are all involved in, which is the technology of derivatives, precisely because no matter how complex the tree that you may have designed in order to price the most complex of derivatives, there will always be the possibility of

writing a derivative on the last derivative that you had. There will also be the possibility of the market of that newly written derivative, by definition, exceeding the tree that you have had so far, because by definition to be trading that extra derivative is going to happen outside your model, because if you had your model why trade it, you would only be evaluating it.

So, again, typically Black–Scholes tells you that the option value is pinpointed by the algorithm. However, the market of options is saying, well, no, now the price is going to be something different and that's, by the way, why implied volatility becomes stochastic. Therefore, there is always the market of the contingent claim, whose payoff you have managed to replicate dynamically, that is always going to be something that exceeds the model, by definition; this is something I have already discussed in *The Blank Swan*. Therefore, now turning the failure of the model positively, we could safely say that what's most interesting in the market – and, by the way, what we can also deem to be the definition of the market – is always that extra step outside of the model. Therefore, we can say that there is no end to the chain of stuff that is going to be traded and admit of prices always outside of the box of their valuation, and once it admits of a price it becomes itself a tradable. So, you can write on it another contingent claim which will itself go outside of the previous box, etc., etc., and this whole chain of things that always go outside the box and trade is what I define now as being the market. You can see now that, by definition, at any stage it will always fall outside of any box, no matter how complex, and this is how it becomes this new matter that I want always to introduce in the void of the previous possibilities.

It so happens that if you think of that through the particular logic of derivatives as we know it, and you find that Black–Scholes gives you ways that you can trade options, as soon as you trade options you go outside Black–Scholes and options became tradable. Then, you move to a higher stochastic volatility model, where now you have, for instance, options as tradable on the market that become themselves the tradable, with which you manufacture a variance swap, say, and then you have a variance swap that you can replicate with the vanillas. But then, guess what? The variance swap itself will exceed its own replication and will start trading at prices that vary from the replication portfolio. In order to expand your space and model those variations you will have to add jumps now to the model that you had, etc., etc., and when you have jumps, you will start trading gap options and the gap options will start trading outside of the box again and again, so this kind of reconstruction is something which I believe is unique to our field. People who are involved in probability theory, and especially philosophers who have thought for all these years about probability theory, did not have it at their disposal and did not think about it.

So, the formalisms of nonarbitrage and finance are always giving me a framework where no matter if derivatives are trading with force in their market, I am always obliged to find another variable that explains their prices, which is the stochastic volatility or the stochastic jump, so derivatives always end up being valued rather than priced with force, and so I'm looking for the missing matter, which is the thing that will allow me to go outside the formalism and create the matter of the trading of the derivatives, which is the same as the trading force in their own market

Dan Tudball: *So, the conventional view of theory and subject, or rather form and matter, starts to come into question in relation to the way the derivatives markets operate?*

Elie Ayache: That's why I think that we have a chance here of thinking of a new kind of formalism that, by definition, will exceed the probability. Then you ask, what is this formalism – can you please show it to me? The answer remains one of my struggles today and I have come to the conclusion that maybe there is no formalism other than that you are using the formalism of Black–Scholes, together with the technology that exceeds it and with the market that you are making with it. So, the formalism ends up being equivalent to the combination of the formalism plus the technology plus the trading, which is a new thing, philosophically speaking. I think that this would not be acceptable in the formal-

ism of physical theories in general, so that is maybe specific to finance here.

Dan Tudball: *What actually provided the justification for this inversion?*
Elie Ayache: I found something very unusual, or very funny at least, even though it was something that everybody knows, or rather may have missed, and this is what I develop in the last part of the book. I found that precisely Black–Scholes may be handing me the opportunity of formalizing this impossible formalism, if you will, and this is something that I explained in Chapter 13 and summarized in Chapter 16 of the new book, and I now believe that this goes back to matter. So, again, you find it's still the same struggle for matter that I want because no matter what we do, the formalisms are only giving us valuations of derivatives, not prices. So, the formalisms of nonarbitrage and finance are always giving me a framework where no matter if derivatives are trading with force in their market, I am always obliged to find another variable that explains their prices, which is the stochastic volatility or the stochastic jump of the underlying price, so derivatives always end up being valued rather than priced with force, and so I'm looking for the missing matter, which is the thing that will allow me to go outside the formalism and create the matter of the trading of the derivatives, which is the same as the trading force in their own market.

I found that maybe there is a door – an unexpected door – that can open itself in the Black–Scholes formalism itself, except that it went unnoticed, and it is the following. I noticed that, not in the original Black–Scholes paper but in the papers that were written after it, in 1979 (by Harrison and Kreps) and 1981 (by Harrison and Pliska), which are the papers that completely formalized and legalized the whole Black–Scholes model under the formalism of martingales and all that. So, especially in the Harrison and Pliska paper, in which you find in the opening paragraphs they say that what they are trying to do here, in a way, is to preserve the Black–Scholes result, except do it in a completely self-contained mathematical theory rather than in the heuristic terms that Black–Scholes did it, more or less, and that's why they resort to martingale theory. What I found quite unexpected, and to me it was an opening, is that if you read what they are saying in their paper – I may be wrong but this is what I found, and I think Philippe Henrotte, our head of theory at ITO 33, agrees with me here – it is that if the underlying asset is trading with force – and we all know what we can do by saying that it admits of its own trading process or stochastic process – it's a price and not a valuation; there are no hidden things that give me the price of the underlying, right? What I managed to do with that is that if you then follow the algorithm of Black–Scholes and make it rigorous, you end up replicating – or they call it "manufacturing for yourself" – a contingent payoff, not a contingent claim. You end up replicating the payoff of a derivative or of a contingent claim that doesn't materially exist yet, and is yet to be written.

Dan Tudball: *So, the real picture is that the option is live and has its own market and is not solely a function of the underlying…?*
Elie Ayache: So, you see the distinction here? That's why I go back to semantics, in order to go back to exactly what the formalism allows me to say and to interpret the formalism without any foreign terms that the natural language I am speaking mixes with the formalism. So, you should go back to the formalism and say to yourself that the only thing that the formalism is giving you at the start is the trading process of the underlying; you have to understand that you are in a world where nothing exists apart from the trading pit of the underlying asset. No derivative exists; how can a derivative exist – has someone on the side of the pit written for you a derivative and told you, please trade it in the same pit? No, because if that's the case, you can ask yourself, where is the trading process of the derivative? Why didn't you give me the trading process of the derivative from the start? No, because, from the start, you only had the trading process of the underlying and no derivative existed. Then, you find that all that you can do is apply very clever trading strategies with the underlying by following the algorithm of Black–Scholes dynamic replication, only trading in this strict world that is limited to the underlying asset pit and only to the trading. You manage to manufacture for yourself contingent payoffs; in other words, you know exactly what the initial premium is that you must invest in order to buy a certain amount of underlying stock. You then have an algorithm that tells you, at every point in time and every price, how to readjust your holdings in a self-financing way in order to end up at the given maturity of your choice with a given amount of stock that if you sold back in the market right then would exactly synthesize the contingent payoff that you had in mind from the beginning.

Dan Tudball: *To some extent, you have solved the chicken and egg question!*
Elie Ayache: This is only what you can do, but it is not a contingent claim; it is only the payoff of a contingent claim. The contingent claim as an independent asset that was written from the start and therefore liable to be traded in its own market has never existed. So, I'm saying that it's *now* that you make it exist. So now, in my reconstruction, it is because you ended up exactly manufacturing the payoff that would be written on it that you create it, and you start trading it; so, in a way, I found that in this reconstruction where I'm following through exactly what the formalism tells you, it's after you are done with the formalism of Black–Scholes and with reading the formalism that the market for contingent claims is created, not before. It's not that the market of options already existed and the Black–Scholes formula only told us a way in which we could trade it in a nonarbitrage way.

Dan Tudball: *The original 1973 Black–Scholes paper was always meant to be a rough-and-ready solution to a problem wasn't it?*
Elie Ayache: That's why in the original Black-Scholes paper of 1973 there is a confusion here because they begin by saying the option exists and its price will depend only on underlying stock price S and time t. So, you may ask, what gives you the right to say that if its market exists, then its price will depend only on the stock S and time t? If its market exists, it will have its own trading forces already pulling it in other directions, so you cannot start up the problem like this. Black and Scholes have smuggled into the assumption something that they will only derive in the end, if you will, keeping in mind that at the end of the argument they will not replicate the option – the option never existed – they will only replicate the payoff of the option.

So, you ask me, what's the difference between the payoff of the option and the written contingent claim, as such? I'm saying that there is a whole difference because if you now consider

the Black–Scholes model, and the formalism it has allowed me, ultimately, if its only result, if its only consequence is to write the contingent claim finally at the end, so it's at the end of the argument that the market for contingent claims comes into being and it is then that we start trading it and if I start trading it, then we are completely *outside* the formalism of Black–Scholes. So, I start again; I reboot the whole system, and only now can I begin with two tradables, which are the underlying and the contingent claim. I start again, with two processes, and I can have a more general Black–Scholes formula – and I'm using the tool to generate a third one, and so on and so on.

But the important thing is that because the contingent payoff is different from the contingent claim, there is an unbridgeable gap between them – you have to write the contingent claim in order to get there. That unbridgeable gap is the void, if you will, because it is outside of the formalism, and it's only when you write the contingent claim that you create new matter, and then when you reset your world and you start all over again, there is no way that this newly created contingent claim can be recreated by valuation again.

All I'm trying to do, because anyone who is listening to me will tell me 'you are only playing on words', it's true in a sense I'm only trying to get the story right in order to ultimately say that derivatives, I've found the way that I can make them tradable by the force of the trading in a formalism, or rather in an extension of the formalism which I have just described. I want to find a way of telling the story right, and I believe that this opening in the Black Scholes argument which was unexpected is what allows me to go outside the formalism and therefore to create this new matter.

Dan Tudball: *So, we can effectively do away with the idea of applying probability because we can replace it with the market of contingent claims… ?*

Elie Ayache: But keeping in mind that I don't want to be understood as saying, as unfortunately some people have understood me in *The Blank Swan*, as saying that anything goes and the prices are just contingent and determined just by contingency, and therefore total chaos. On the contrary, as you picked up on in the notes, I don't want the view from chaos, where indistinctly, if you will, derivatives and underlyings trade in the same arena. Of course, ultimately, I want to get there and say that there is no difference between them, in the sense that the derivative is not valued but is also priced – there is no difference, in that both of them are in the market and admit of pric-

Black–Scholes Redeemed

Ayache's reinterpretation of Black–Scholes is at the heart of *The Medium of Contingency*. Here, the author gives a quick guide to getting to the heart of that argument.

It is simpler to say that BSM has, in fact, only something to do with the underlying asset and its trading. From inside the trading pit of the underlying asset, there was never mention of writing options or generally contingent claims on that underlying. We do not know what writing a contingent claim means, when our sole business is to trade the underlying. If anything, writing has to be invented; writing engages the conceptions of a delay in time (the maturity of the contingent claim) and a delay in space (the strike or generally the conditional provision of the contingent claim), and these conceptions are not available to anyone trading the underlying asset and focusing only on this. It is only we, when we speak our natural language, which is 'looser' than the formalistic language, who adopt a loose conception of the 'market' and of 'time,' and who wrongly believe that this loose concept of the market can include both the underlying and the derivative. This loose concept of time can include both the time in which the underlying is strictly traded (the time inside the trading pit) and the time in which the writing of contingent claims can be conceptualized. If we want to be strict about the formalism – and this is what I am pursuing in the book – we have to be strict about the market and about time: hence, strictly speaking, when all we have is the trading of the underlying asset, which is summarized by a certain volatility sigma (in other words, when all we have is the assumptions of BSM), we cannot conceive of the writing of the contingent claim or derivative, let alone of its trading.

Chapter 13 of the book is where I express these things best. When BSM is looked upon as it should – that is, from the strict point of view of the trading pit of the underlying asset – we find that it is a *complete* characterization of that trading. Indeed, BSM is not just equivalent to giving the underlying stochastic process (Brownian motion), but also gives dynamic trading strategies involving the underlying. BSM is the conjunction of the random underlying price (described by Brownian motion) and of the dynamically adjustable size of the holding of that underlying, in a self-financing way. Obviously, such self-financing trading strategies of the underlying asset end up producing, at their maturity, amounts of money that will depend on that maturity, on the price of the underlying that prevails then, and, more generally, on the whole path. This is true of all stochastic processes and not just Brownian motion. Now, it so happens that Brownian motion has the miraculous property that, instead of running just any self-financing dynamic trading strategy and ending up with the amount of money that will just be its result, we can fix the amount of money beforehand; that is to say, we can predefine the amount of money that we wish to end up with, at maturity T and at the underlying asset price S, as a function $f(S, T)$ of that maturity and underlying price, and control exactly (depending on the actual path that the underlying will follow) the dynamic trading strategy that will end up producing that final amount $f(S, T)$, no matter the path followed. In other words, under Brownian motion, we can manufacture any contingent payoff $f(S, T)$ that we may have defined beforehand, no matter the path of prices of the underlying asset, just by holding, at any intermediate time t, and price S_t, the adjustable size delta(S_t, t), which is known as the BSM delta.

This is how writing is invented. For, when we realize that we can manufacture the contingent payoff $f(S, T)$, no matter what, this becomes equivalent to writing it in advance, under the form of the contingent claim, or written contract. This does not mean that

es; however, it is not chaos because I definitely need the Black–Scholes algorithm in between. I'm not at all dismissing probability – this is what the philosophers who have read *The Blank Swan* and who love contingency have thought, that I was dismissing the whole field; no, on the contrary, more than ever here, in three different instances that I can talk about later, more than ever I insist that all I'm saying in this book relates to derivatives that admit of a tradable underlying. Otherwise, what I say doesn't apply – I'm not talking about weather derivatives or CDOs or such like, I'm talking about derivatives that you can hedge continuously with the underlying, and it's from these effects that I will deduce this matter of the market, if you will...

Dan Tudball: *So, one part that I wanted to confirm with you was whether … theories or models that arrive at a stochastic process for market price do not assimilate trading … is assimilate the correct word? The process of trading in these formal models is not something that is entirely assimilated into these models; we see trading as something that produces the stochastic process and then is screened off from that and we are just looking at data being generated in a random way, but the actual process of trading – what you call the force of trading – is not assimilated. Now, can we focus a bit on that…?*

Elie Ayache: Yes, absolutely – and, by the way, that's the key concept in the whole book, if you will. Again and again, my preoccupation is to argue that as necessary as stochastic calculus and stochastic models may be to us as tools, they will miss the force of trading, which is the trading as I knew it when I was immersed as a trader inside the pit. All I'm trying to do is tell, if you will, the best philosophical narrative or the best philosophical story that can restore the force of trading through the formalism. That's my purpose really – that's what I'm trying to do.

So, now, of course, we all know that as soon stuff trades freely in an exchange, it has to be random, just by the efficient market hypothesis, because it cannot be predictable at all; if it were predictable, just people rushing to arbitrage it away will make it trade at that price instantly and any further price will have to be random. So, definitely, as soon as you trade something it is going to be random, but the thing is to argue that there are two different registers of time here – two different perspectives.

So, to repeat, for somebody who is immersed in a trading pit trading, of course he is generating randomness, and even more than this he is actually in direct contact with the event itself, he is in contact with the

the 'newly created' contingent claim is traded yet – at least, not in the same picture. Trading it and gaining access to its market is an extraordinary step that will take us outside the previous picture and formalism. This is not a sequential process happening in time, mind you. It is not that we can now trade the contingent claim, *after* we have manufactured the payoff that is encoded in it. From inside the trading pit of the underlying, we still have no notion of anything else being traded. It is here that I introduce a major idea, which is that, from inside that trading pit, we also have no notion of probability states, of statistics, or of time series. From inside the trading pit, there is only the trading force. Trading and its force are, as a matter of fact, primitive to anything else, and by that I mean, primitive both to the notion of a stochastic process *and* to the notion of writing and trading a derivative. However, the trading force of the underlying has to express itself one way or the other. We have to exit, one way or the other, from the formalism of trading the underlying, in order to interpret that formalism in reality, or in order to append matter and reality to that formalism.

One way of exiting is through the stochastic process and the time series of prices of the underlying. This results in statistical studies of the underlying asset price and in the *valuation* of lotteries written on that random generator – lotteries which everyone confuses with derivatives – and a valuation of lotteries – which everyone confuses with the pricing of derivatives. However, this kind of exit in time series and statistics is definitely an exit *outside* the market. Simply, the trading pit of the underlying ends up being assimilated with a random generator. An alternative exit consists in blocking the exit in time; that is to say, it consists in withholding time (astonishing as this may sound) and in exiting in place instead, or in the writing dimension, or in the materialization of the contingent payoff into a written contingent claim; that is to say, it consists in saying that, instead of the trading force of the underlying asset expressing itself in time, and of admitting a volatility coefficient which measures its process in time, it can express itself directly in place, in the price (not the value) of the contingent claim or derivative that has just been written. The major discovery, here, is that *implied volatility*, which is what we compute from the price of the contingent claim, takes place in the totally alternative exit to 'real volatility.' This is why the two are incommensurable to each other. To repeat, my reasoning does not take place sequentially in time (trading of the underlying, manufacture of contingent payoffs, writing of contingent claims, trading of contingent claims) because time is only one possible dimension. It is important to realize that giving the price of the contingent claim, or giving their market, is an alternative to giving time, and therefore is as momentous and important as giving time. It cannot be deduced any more than time can be deduced. It can only be decided, or chosen, as one possible exit and interpretation of the formalism, or as matter to append to the formalism (the formalism of the trading force of the underlying asset). Of course, it has the advantage, over the exit in time, that it doesn't exit from the market. To the contrary, it gives the total view of the market, what I call the *reality* of the market (or the creation or genesis of the market) – the one that is completed with the derivatives prices. Chapter 16 provides a good summary of all this.

Now that the price of the contingent claim is given (to repeat: at a total disconnect with the time process of the underlying asset price, which can only produce valuations of derivatives lotteries and no market), now that it is given anew in a total leap into the void, the whole process can be reiterated. Now, the contingent claim can itself act as an underlying asset, whose trading pit is considered afresh and whose trading process is given, and such that contingent claims can be written on it, in turn, after contingent payoffs are manufactured, etc. Again, the reasoning is not sequential in time. This continual reiteration of the formalism of BSM, and of the exit from it, happens all at once. However, every time we introduce quantitative tools, we have to make a cut somewhere and consider a section of that reality; this 'temporary stage' results in calibration and recalibration.

incalculable event, but as soon as you want to model that mathematically from outside, you convert that result to a stochastic calculus and prices as seen from outside cannot but look like a random generator's time series of numbers. That's obvious because – I mean, I'm not challenging that – you have no other way of modeling things from outside, other than by stochastic processes. However, what I am trying to say is that fine, however we are offered the chance with the rest of the story, which is the dynamic hedging following Black–Scholes and understanding the semantics of that, understanding what really is happening there and how this is actually now creating the capacity of making the markets in derivatives, I'm trying to argue that because of that there will be an excess of the whole stochastic statistical picture and this excess can be reinterpreted as giving us back the force of trading that we have been missing as soon as we stepped out of the pit to model it.

I reboot the whole system, and only now can I begin with two tradables, which are the underlying and the contingent claim

One of the problems which every quant knows that I'm trying to tackle is that we all know that the basis of the Black–Scholes theory is that the underlying itself trades in its pit, so you have the force of trading here because nobody is actually discussing why the stuff is going up or going down – it is simply because people are trading it and because you are only buying it, if you will, in anticipation of the price going up; you are grabbing it because you believe that it will go up, and by grabbing it you are making it go up, so that's the force of trading. Everybody knows that. However, the main problem is that when it comes to derivatives, you can now value derivatives through the stochastic process that this generates, and when you value derivatives, you end up with formulas like Black–Scholes that give you the derivative value and therefore it is a valuation – it's no longer the force of trading – and, as a matter of fact, all quants know that we are in trouble trying to model a market for derivatives.

If you ask me, or ask Philippe, or ask anyone here how do we model the market for the underlying, everyone knows how we do it: we just write a stochastic process and say that the underlying asset has some volatility, sigma, etc. But now, how do we model a market that is alive with the force of trading for its derivative? Now, you have two ways of doing this. If you want to do this in a kind of phenomenological way, you have to do what Rama Cont has said in one of his papers, that people who are trading the derivative with force in the trading pits are buying and selling something which is the Black-Scholes implied volatility. When we are in the trading pits, we would say that I want to buy implied volatility, which means that I want to buy options, no matter what strike or what maturity – I will decide this later, depending on how I want the greeks to evolve over time – and that's why options are more complex to trade than the basic asset. Nevertheless, the basic concept is that I want to grab volatility and buy it because I expect this to rise, and maybe my own buying action will make it rise, by the way, because I'm pulling the market or because I want to sell the implied volatility. Now, that's what happens in reality, but then I say, go ahead, please, and model this for me, so you have people who have tried directly, in parallel with the stochastic process that I have written for the underlying asset price, to write a stochastic process for the Black-Scholes implied volatility, considering implied volatility as the basic commodity that option traders are grabbing or selling in the option pit.

Typically, you have a two-dimensional model, one for the underlying which is still trading with volatility, and in parallel to that you add another process for the implied volatility of some other option, but you're not done because you still have to link the two in a nonarbitrageable way, so what you find is that you cannot stop here; if you want to have a model that you can use as a technology and then use to trade and have nonarbitrageable prices for stuff, you cannot stop here and you have to make the assumption that the volatility of the underlying is stochastic and you have to then work out by very hard computation and by mathematical derivations, to work out very hard and try to infer what formally the stochastic process of the volatility of the underlying price should be in order to be compatible with the stochastic process of Black–Scholes implied volatilities that you have given yourself from the start.

So, you start out by saying, I'm going to model things as they look from inside the pit, meaning forces of trading for the underlying, and forces of trading for the derivative and forces of trading for the derivative you can only first of all write as some process for implied Black–Scholes volatility; then, in order to make sense of the two, quantitatively speaking, and to be able to produce a model that does not generate arbitrage opportunities, you have to really then interpret the Black–Scholes implied volatility as the implied volatility of the particular option that you have selected by the way and then to link that to explain the fact that implied volatility is changing therefore to explain the fact that there is trading in the option market in the first place. This is because the volatility of the underlying – by which I mean the real volatility of the underlying asset – is stochastic, except that you will find that it will have to follow a very complex process in order to make sense of the process that you have written for implied volatility. So, all I'm saying is a simple fact that every quant knows, meaning no matter what you do, if you want to get something formal and something that you can use without arbitrage, you will never be able to model the price of an option or of a derivative as an independent state by itself, if you will; you will always have to find a hidden state that the volatility of the underlying is stochastic as if by nature, and as if that stochastic nature of the volatility of the underlying was the cause and the explanation for why there exists a market on derivatives.

Dan Tudball: *You end up a long way from the truth…*

Elie Ayache: You end up with a formalism that doesn't reflect reality, which is that, in reality, the option market is obeying its own trading

forces and option prices are going up or down, even when the underlying is not moving, just because people are grabbing them in their pits. This observation is simple, but I believe it has not been resolved until now, and I'm sure it cannot be resolved formally if you want to apply nonarbitrage principles – and we all want that because there's no way I am going to sell technology to my customers when the prices that are produced by my pricing tool do not verify nonarbitrage. You cannot *not* verify nonarbitrage. There is no way that you can have that, and have as the floor level the one where the options trade on their own, in their own market, with their own force. You have to go under that floor level to discover the hidden variable, which ultimately relates to the underlying alone, such as stochastic volatility or stochastic jumps, or whatnot, so that is something everyone will agree about; however, it seems to people that there is not much you can do here, so they just leave the problem as it is. What I'm saying is that I'm still philosophically not happy because I want to formalize the market of the contingent claims or the market of the derivatives, and formalize the fact that it trades with force, even if that means I have to go outside the traditional formalism of probability. This is what I'm trying to achieve.

Dan Tudball: *So, the key is, we should retain volatility as known and constant in BSM, in order solely to produce the contingent payoff, but discard the step that implies (what ultimately will become) stochastic volatility from the price of the traded contingent claim and feed that backward through the equation? Does this mean that the only volatility figure we are going to use in calculating the contingent payoff in BSM is (constant) historical volatility of the underlying price?*

Elie Ayache: Consider this. Volatility is known and constant in BSM because BSM is a formalism. It is not in reality that volatility is known and constant; it is 'known' and 'constant' as a symbol in the formalism. Before volatility starts to vary numerically, we have to decide how the *symbol* should vary – or, in other words, how we should vary from the formalism. Again, two exits present themselves. Either we vary from the formalism in the time-series dimension: volatility becomes the statistical coefficient that it is, we start observing time series of underlying prices and time series of volatility, we observe stochastic volatility, we imagine random generators that are more complex than Brownian motion, and we start evaluating derivatives lotteries under general stochastic processes, and notice that there are now many consistent valuations (i.e., respecting nonarbitrage), many equivalent martingale measures, etc. But these valuations are no market. Or, we vary from the formalism in what I call the place dimension, which means that we consider the prices of derivatives as given in their market, or, again, we consider the existence of the smile. The existence of the vanilla smile is simply the acknowledgement of the fact that both the underlying asset and its first-level derivatives now trade and have their trading processes as given in the same place. We calibrate a generalized BSM to their prices in order to reproduce the next iteration in the 'formalism-exit' sequence, in order to manufacture the contingent payoff of the next level and write the contingent claim of the next level (in this case, the exotic option), in order to bring into reality the price of the latter, etc. This generalized BSM will have to be, of course, some kind of stochastic volatility/jump diffusion model, which only serves the purpose of generalizing the self-financing dynamic strategy that will replicate the next contingent payoff. We believe the regime switching is the best framework because of its potentially endless recalibration capacity and the potentially endless number of regimes.

So, the step where we manufacture the contingent payoff is still inside the formalism of the trading pit of the underlying asset – it is equivalent to its trading force, we said. In this step, volatility is constant because it is formal, it is a symbol. Neither the *reality of time* (which will make it vary in time and become stochastic in time) nor the *reality of the marketplace* (which will produce volatility smiles and make volatility vary in the model, if only to be able to calibrate the volatility smile) has yet got a hold on it. Historical volatility has nothing to do with any of this. Historical volatility has nothing to do with either the formalism or the two exits from it. We should not confuse historical volatility with the 'real volatility' that we get once we exit the formalism in the time dimension. In the time exit, of course, volatility becomes quantitative and is no longer a symbol (i.e., qualitative); however, it is conceived as instantaneous: in theory, we could estimate it in a single instant, and not over a finite period of time or history, because of the marvel of Brownian motion, which allows us to consider an infinity of random samples of underlying prices, yet have them all compressed in a shrinking interval. In the 'place' exit, volatility also becomes quantitative and is no longer a symbol; however, it is immediately understood as implied volatility, which is the other name of 'market price of a certain derivative.'

In my view, which is the 'place view' or the 'market view,' the time dimension shouldn't be considered anyway. You may only consider it if you exit the market and start indulging in statistical studies of the underlying price, and in evaluating lotteries written on that random generator. How you estimate volatility, in that case, is a different problem. It is a practical problem. Using historical volatility is a strong assumption (as it assumes stability of the random generator, etc.). But if you follow me and choose the place view or the market view, there is no time anyway, and all we have is the formalism and the exit from it as calibration to the smile; that is to say, all we have is recalibration (not even conceived in time, but along the endless chain of derivative writing).

In short, there is no 'volatility figure' in my view, except implied volatility, which is a transient figure anyway as it is always already taken over by recalibration; that is to say, always already taken over by the set of implied parameters of the next instance of the regime-switching model (implied volatility being the first instance of calibration, ever, when there is only a single volatility regime).

But now, how do we model a market that is alive with the force of trading for its derivative?