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God’s Model vs. Market Models Part II: The Importance of a Book

Two fundamental concepts of Bergomi's, the *trading decision* and the *pricing function*, turn probability merely into an interpretation.

In the previous episode, I mentioned the importance of Bergomi’s book (*Stochastic Volatility Modeling*), over and above Bergomi’s model. From the start, he meant it as a book on market models, in which to build the case for them. Precisely, the path the book follows is a build-up, starting from Black–Scholes–Merton (BSM), in which the inaugural distinction between pricing equation and pricing model is made, and ending with the forward variance models, with a crucial passage in the local volatility model and the elimination of Heston along the way. The local volatility model is the simplest market model, according to Bergomi, and this is why it is crucial in establishing the concept. This is all the more so as it is, in theory, a model in which vanilla options are perfectly replicable with the underlying, hence in theory redundant and incompatible with the idea of their independent market. Precisely, the inversion of causality, characteristic of Bergomi, will be nowhere as decisive as in the local volatility case. It is such that the vanilla options will never be considered an output of the model, but are always an input. A crucial notion is the hedge ratio to apply to the underlying asset, in whose calculation the vanilla option surface is held fixed. This means that as the underlying asset price is changed, the local volatility function must be recomputed in order to match again a vanilla option surface *that hasn’t changed*, and then the change of the exotic price finally computed under this new local volatility function so as to complete the computation of its partial derivative with respect to the underlying price. To be fully consistent, Bergomi almost immediately proposes an alternative in which the price processes of vanilla options are given

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in all generality, each one with its own volatility and driving Brownian motion, before specifying the conditions to make it collapse to the local volatility model.[1] This becomes a general method characterizing the market models: modeling the implied volatilities, either of the vanillas or later of the forward variance contracts, directly. Since the pricing formula (no longer a model) is now only a black box,[2] it becomes irrelevant how its internal machinery reacts. In the case of the local volatility model, the local volatility function $\sigma(S,t)$ happens to vary, as the vanilla implied volatility surface varies under market forces, and it is recalibrated to that surface. However, in the forward variance model, two successive market observations of the forward variance curve do not change the internal machinery; they simply act as two successive spot observations of the underlying variable, and the model is designed to fit it by construction. Bergomi’s point is that the local volatility model is no different. It is primarily a market model, and the full vanilla surface in it is also spot. As a matter of fact, Bergomi (2017) later combines the two models.[3] He now considers a local stochastic volatility (LSV) model, driven by a forward variance model. The full vanilla option surface is now again spot, and the vanillas are now again the hedging instruments; as for the forward variance curve, it becomes part of the internal machinery, together with the local volatility function $\sigma(S,t)$. We no longer care in what way the two will move to accommodate two successive spot observations of the vanilla surface.

A crucial notion is the hedge ratio to apply to the underlying asset, in whose calculation the vanilla option surface is held fixed

Bergomi’s whole book is meant to inscribe the forward variance model (which happened to bear his name) in the faith of the market models, as opposed to any other religion or God. He wouldn’t have gone to great lengths to write the chapter on the local volatility model if such was not precisely his intention. To repeat, market models are not models of the underlying asset price. By that, I mean they are not stochastic models, or models with stochastic structure. To put it even more bluntly, *they do not involve probability*. Probability, expectation, stochastic structure, and stochastic processes, these are notions associated with models of the underlying asset price alone, and their role in the market models reduces merely to an interpretation. Proponents of underlying stochastic models can try and fool you, and present them as models of the market. However, what they are really are models in which trading is not truly involved, but only simulated. And what their authors prepare to do, is evaluate derivatives as expectations of some kind (the famous martingale measure of the market), not price them. Market models start properly when derivatives written on the underlying asset are given as trading from the start. This does not mean that stochastic processes are written for them (these will only arise later as a mere interpretation). Rather, market models start with a *pricing function* in which the prices of vanilla options are simply arguments.

The trading decision and the pricing function

A *pricing* function whose arguments are *prices*: this is the fusion formula or the code that is definitional of trading and of the market, in Bergomi. When the prices of derivatives are considered as state variables, this means, by our lights, that there precisely will be *no stochastic structure*. Something Bergomi calls the *trading decision* is made, instead. The pricing function, together with its pricing arguments, which is what trading means in Bergomi, is inseparable from the *trading decision*.[4] Stochastic structure was needed to evaluate derivatives by computing their arbitrage-free value as expectation. However, when vanilla options are supposed to be trading already, as the hedging instruments of further exotic structures, we no longer *evaluate* the exotics; we price them by using the pricing function whose arguments are the prices of the vanillas. Evaluation under the stochastic process (by expectation) is replaced by the evaluation of a peculiar function, whose output is no longer a value, but a price.[5] It is not some stochastic structure that yields the pricing function, as if the exotic option was derivative on the vanilla options. The pricing function is given; it is the market's. The vanilla options prices will keep underlying it until the maturity of the exotic option and beyond, and all we ask is that the hedged position composed of the vanilla options and the exotic option should have its profit and loss (P&L) under control. The assumption that the prices of the vanillas be given is the assumption of the market models (it is the assumption of a market for vanillas); it is what replaces the stochastic structure and all the arbitrage worries associated with it. This is the key element in the philosophy of market models, which is so hard to understand. If the prices of the vanilla options (or of the forward variance contracts) are the primary state variables, given by the market and by nobody else, then they can no longer qualify as outputs of a calculation, typically an expectation, and they are no longer underlain by a stochastic process. Such an underlying stochastic structure, together with the logic of valuation attached to it, no longer exists. For this reason, the exotic option cannot be said to be underlain by a stochastic structure either; it is, strictly speaking, no longer a derivative (either on the underlying asset or on the vanillas), because no derivative exists anymore and the whole logic of underlying and derivative is abolished. Only the surface of prices remains, and the exotic option price is just a function of the prices of the vanillas (or generally of the hedging instruments). As we will explain later, the logic now flows forwards and no longer backwards.

A *pricing* function whose arguments are prices: this is the fusion formula or the code that is definitional of trading and of the market, in Bergomi

We've always been suspicious of BSM's first postulation, which is that the option value $V(S,t)$ should be a function of time and of the underlying asset price S alone. Why is it not additionally a function of something else, $V(S,\ldots,t)$, a variable specific to the options market? Why did BSM condemn the options market before its birth? Was it because, historically, they had no way of guessing that options trading would be the main consequence of their formula? Couldn't they have anticipated this simple fact of logic, and allowed into their option pricing formula the variable that options prices will be additionally and even crucially a function of, namely an implied volatility index or identifier of some kind, thus preventing the pricing formula from breaking itself systematically from outside, by the very law of its usage, and therefore relieving us of the smile problem from the start? In voicing this complaint, we unconsciously put ourselves in the logic of market models, which is foreign to the original BSM. We import into the origin of BSM a worry and a question which only emerged later. For BSM, once again, were proposing a value, not a market price, and they were working under a stochastic structure. The *trading decision* was not available to them. The stochastic structure was their only ground, and it left nothing for decision; it imposed the format of the option value and all that remained to do was to make that format explicit. Stochastic volatility models that came after BSM, such as Heston, sought to generalize the stochastic structure (or the underlying process). The value of vanilla options now explicitly depended on volatility as an additional state variable. Perfect dynamic replication with the underlying asset was replaced by optimal dynamic replication (typically, in the mean-variance sense), following an argument from stochastic control. But two different stochastic volatility models, say with a different number of factors, could now perfectly agree on the vanillas spot market prices, yet disagree on their (optimal) hedging strategies, or correlatively, on the prices of exotic options (typically barriers or cliquets).[6]

This meant that the derivative pricing model had to be calibrated to the exotic options market, on top of the vanilla options market, in order to better hedge the vanillas, or in other words, to better price them! In turn, the hedging of the exotics depended on the prices of more complex exotics still, a peculiarity which meant that our original pricing model $V(S,..., t)$ would, in the end, not only depend on volatility, but on the volatility of volatility, now a state variable in its turn, and on the volatility of the latter, so on and so forth.

This infinite regress comes about because we are trying to generalize the stochastic structure and to determine which is the true and ultimate one,[7] instead of moving to market models and a *trading decision* being made as a result. The trading decision breaks the chain from the start. "Calibrating a model," writes Bergomi, "amounts to *deciding* which (vanilla) instruments our exotic option price is a function of, along with the spot. The consequence is these instruments are our hedge instruments" (p. 234). Emphasis is put, once again, on the word 'decision.' Of course, the prices of the vanilla instruments we chose as hedging instruments are not a function of anything, neither of volatility nor of volatility of volatility. They are given by the market. Also, we now appreciate the double-faced character of the trading decision. Not only do we decide which instruments will be our model's underliers, but we decide to limit the underliers to those instruments precisely (the vanilla options or the forward variance contracts, in this case). Recall that Bergomi had lamented the fact that a market for the volatility of the prices of his vega-hedging instruments was not (yet) available, a matter which had stopped him from modeling the implied volatilities of options written on his forward variance contracts, on top of modeling the implied volatilities of the latter, and from extending accordingly the range of hedging instruments to the next level (p. 220). Precisely, the trading decision was meant to limit the arguments of the pricing function to the prices that were available, in this case the vanillas. Recall that this pragmatic disposition (and decision) is possible in the minimalist philosophy that Shafer and Vovk have described, and is essentially allowed by the ex-post logic of the accounting equation (instead of the ex-ante logic of the structure): deciding to play the game of controlling the P&L of the hedged position only with so many prices the market is showing me and no more.[8] Stopping the game at his level, deciding that a break-even level of volatility $\hat{\sigma}$ will be possible when the underlying asset is the only hedging instrument, or that a break-even covariance matrix will be possible with the hedging instruments we select additionally, is what enables Bergomi to close the partial differential equation (PDE), and to derive the pricing function. There is arbitrariness, or rather, there is a *decision* (which is an open process, as said), in the way the PDE of the market models is closed, whereas the PDE is closed, in the traditional logic, by the mere hypothesis of structure, which leaves no choice and closes everything automatically.

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Closing the PDE by the argument from accounting (the game), rather than by the fundamental hypothesis of stochastic structure, is essential to making this structure come second in the reasoning, and therefore to turning probability and all the subsequent underlying stochastic processes merely into *interpretations* (and consequently relaxing the non-arbitrage constraints). This is essential to the philosophy and nature (and standing) of the *pricing function*. We know that a parabolic PDE admits of a probabilistic interpretation, and it is because of the PDE which he derives first from an accounting argument that Bergomi is able to write stochastic processes for the hedging instruments, never failing to remind us that these processes are only probabilistic interpretations (p. 219). Even better, it is now essential that these should only be interpretations and not a real ground for valuation. Why? Because the probabilistic interpretation of the PDE means that the price of the exotic option is the discounted expectation of its payoff under some measure, no matter whether real or not. Since its payoff derives solely from the underlying asset price, this means that only the underlying asset price, and parameters that affect its stochastic process (i.e., volatility or volatility of volatility) can affect the exotic option price. Conditionally on the prices of all hedging instruments ξ (other than the underlying asset) being fixed, the option pricing proceeds backwards in a tree, as the calculation of an expectation where the underlying asset price is the only variable . That calculation would have no reason to change, along the orthogonal direction of the price of another hedging instrument (i.e., the direction of another state variable ξ), if moving in that orthogonal direction did not affect the expectation calculation in one way or the other (i.e., typically widening the spread between S_u and S_d or changing the probability weighting – to project ourselves in the binomial case); that is, if it did not affect the parameters of the underlying process. This seems to suggest that the implied volatilities of the hedging instruments that Bergomi is modeling directly are merely proxies of the instantaneous volatility of the underlying asset price and that we are back to the remapping.[8] But we know this is only an interpretation and the underlying process does not exist. The prices of the vega-hedging instruments seem to affect the exotic option price, when they vary, in the same way as the volatility state variables in the traditional approach would affect it, and it is even the case that the short-term forward variance ξ_t^t is equal to the instantaneous volatility σ_t of the asset; however, I insist that the latter is, also, only an interpretation, and there is nothing, in the way that the market models are presented and the *trading decision* is made, to compel the hedging instruments to have anything to do with the underlying asset, let alone to be derivative on it.

There is no financial theory or arbitrage theory in the market models. The implicit assumption that replaces them is that the market has solved for us the relevant problems. We are not in the process of creating the market or explaining it, or showing that its prices are arbitrage-free. We just assume that the market is given, that it exists and has existed long enough for, somehow, the prices it is assuming for the hedging instruments (or our future state variables) to be already arbitrage-free, or at least, to have been out there long enough for there, obviously, not to be free money to be made anymore. Bergomi explicitly says that the configuration of prices of the hedging instruments to be used in the market model should not be nonsensical, which we interpret as being arbitrage-free (p. 25); however, he insists all the time that they are state variables (i.e., totally free, never to be constrained by the covariance structure of any model we should later use in practice). Of course, the practical model (or temporary structure) we are using will automatically implement non-arbitrage, and of course prices of the hedging instruments will ultimately be recovered as expectations under the underlying stock process, if only because the pricing PDE has a probabilistic interpretation, and the reduction to the finite practical model is only one way of computing the PDE. However, philosophically speaking, the given market prices of the hedging instruments are already arbitrage-free *by definition of the market as being given and persistent*, and are not arbitrage-free because some valuation model imposes this condition. Market models assume a lot has already been solved by the market or is implied by the market. Not only is financial theory and arbitrage pricing replaced by a simple pricing formula, whose output is deemed a price as long as the hedged position may break even,[9] which means that the existence and persistence of the market has internalized the fact that the only purpose of pricing by arbitrage was ultimately the control of P&L, but the market's existence and persistence, or the market's history and practice, *is the only warrant that the hedging instruments are relevant*, or that they even affect the price of the exotic option to begin with. Bergomi keeps referring to the option trader (Nazim Mahrour) who first articulated to him the market models principle: "Options are hedged with options" (p. 15). This revelation occurred, by Bergomi's own confession, at the very start of his carrier in finance, and explains why his thinking always starts with an existing options market and not with an underlying stochastic structure. Given how he presents the problem in all its generality (i.e., the hedging instruments are anything whatsoever), we may wonder what their relation to the exotic option may be. It is almost in passing that the suggestion is made that they are here to offset the underlying gamma risk; which means that they must be derivative on the same underlying ultimately. I say 'it is almost in passing,' because for the rest of the book they will come to be recognized as vega-hedging instruments, in charge of hedging nothing else than the changes they themselves cause in the pricing formula by simply figuring as its arguments, and not as gamma-hedging instruments, a circularity that formally insulates them from the underlying asset. We all know that the volatility state variables are the ones that affect the price of the exotic option in the traditional approach; however, in Bergomi's presentation, all that is required is the existence of a covariance break-even matrix involving all the hedging instruments, including the underlying, without any distinction (so they could be any instruments, even not written on the same underlying, possibly only correlated with it), and it is only through the expansion of the probabilistic interpretation (as above) that we understand that the space they span must be a space where parameters of the underlying process (e.g., volatility) implicitly vary.

In the first instance of hedging the gamma of the exotic option with the gamma of the vanilla option, where both gammas are computed using the BSM formula, Bergomi emphasizes the fact that the volatility to use in BSM (for both options and both gammas) can no longer be the estimate $\hat{\sigma}$ of the realized volatility of the underlying asset price

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at the end of the day, as was the case when hedging with the underlying asset alone, but has to be the implied volatility $\widehat{\sigma_0}$ of the vanilla option used for hedging the gamma (p. 16). This is how the concept slides from realized volatility to implied volatility, and the P&L is no longer called 'carry P&L,' but 'marked-to-market P&L.' When we were trying, in the first pass, to justify the usage of the BSM formula and not model, realized volatility was the key parameter, and this is how the market, or the quant, was supposed to hand us the pricing formula of that option (no matter whether vanilla or exotic). However, now, with the assumed existence of a volatility market (and this is key – i.e., the availability of the vanilla option price), we no longer need to justify BSM as a pricing formula; we are no longer making the vanilla option price; its price is given, and we can now use BSM as a formula, for both the vanilla and the exotic option, in order to cancel their gammas (notice that we are still discovering the price of the exotic option; its price is not given). The problem has shifted. We no longer are in a genesis or discovery mode concerning the price of the vanilla option because this price is given. Now, BSM is just an arbitrary black-box formula. We use it to imply the vanilla option volatility, then we use it with that volatility to compute the exotic option gamma and the ratio of vanilla options needed to cancel that gamma. This is not saying that BSM will be the pricing formula of the exotic, for our total P&L will now depend on the price movement of the vanilla option. This whole proto-reasoning is only here to show that we no longer depend on the realized volatility of the underlying asset price (as gammas are now canceled), but instead on the market price of the hedging option, therefore on the realized volatility of *that* price. This is now called vega-hedging as we are now sensitive to changes of the BSM implied volatility of the vanilla option, and we are interested in the realized volatility of this implied volatility. The only problem is where to express the volatility of implied volatility. Bergomi rightly notices that it is not within the reach of BSM, hence the recourse to stochastic volatility models, which, to insist, have no other purpose than expressing the volatility of implied volatilities of the additional hedging instruments.

This is not saying that BSM will be the pricing formula of the exotic, for our total P&L will now depend on the price movement of the vanilla option

The proto-reasoning temporarily eliminated gamma risk, or the gamma–theta P&L break-even equation relative to the underlying asset, and evidenced the dependency relative to the implied volatility of the other instrument; the problem moved away even farther from the existence of the underlying process and the corresponding register. However, going beyond the proto-reasoning and finding the stochastic volatility model will now reintroduce gamma–theta P&L for all the hedging instruments, in equal measure, including the underlying asset. This will not stop Bergomi from using the expression 'vega-hedging' in the stochastic volatility model in which vanilla options (or other derivatives) are additional hedging instruments, or new underliers, despite the fact that the P&L is now again carry P&L and break-even concerns gamma–theta again, and requires break-even realized volatilities (or covariance matrix) again.

The tacit presupposition is simply that if this were not the case, the pricing function would not hold long enough in the market, and more particularly, would not have held long enough already

The whole purpose of the proto-reasoning involving the vanilla option is to introduce the *trading decision*, which is not related to financial theory or derivative valuation theory any longer – the simple decision to make the exotic option price a function of the vanilla option price. This becomes the starting point of the theory (and whole philosophy) of market models. The proto-reasoning convinced us of the necessity of a second hedging instrument (the vanilla option, to cancel the gamma), and from then on, we will simply *postulate* that the pricing function of the exotic option is a function of the prices of other derivatives. No justification is given as to why we need to hedge using the full vanilla surface and not just one vanilla option (or later, the full forward variance curve). Simply, the project is now to model the independent dynamics of those additional instruments, and yes, the whole idea is *to forget that they are derivative on the same underlying asset*, for that would reintroduce the underlying stochastic process. The whole difficulty of the exercise (the paradox somehow) is that Bergomi insists the additional hedging instruments are totally independent (they are state variables on the same footing as the underlying asset), while they cannot ultimately be anything whatsoever because they are here, in the end, to hedge against risks that emanate only from the underlying asset. The solution of the paradox is to *forget* the second jaw of the paradox and to maintain that the price of the exotic is *simply a function of the prices of the hedging instruments, and will forever continue to be so*. We know that the hedging instruments are not anything whatsoever and must ultimately be derivative on the underlying asset; however, this knowledge is no longer part of the formalism – this is now hidden, implicit, or taboo. The tacit presupposition is simply that if this were not the case, the pricing function would not hold long enough in the market, and more particularly, would not have held long enough already. So there definitely is a sense of something immemorial, in the market models, something – a memory, a genesis – that may have existed in time immemorial but has been now completely taken over and replaced by the market.

From backward to forward

Yes, I must admit that Bergomi has solved the problem that had long arrested me, that of having ultimately to relate the dynamics of the hedging instruments to the underlying asset process, on pain of arbitrage.[10] There is no non-arbitrage principle any longer in Bergomi, not in that sense. It is all absorbed in the mystery of the control of the P&L. And yes, from start to finish, we are modeling the implied volatilities of the hedging instruments, as resulting from the proper forces of their market, and their market price dynamics has totally replaced the dynamics of the parameters of the underlying process. We no longer model stochastic volatility, but model only the volatility of price of (say) a variance swap, because of the trading decision: because we have *decided* that from start to finish the price of the exotic is a function of the price of that variance swap. It is only the probabilistic interpretation that made us think that the variations of market prices of the hedging instruments were in reality hiding the variations of parameters of the underlying process, such as volatility and the volatility of volatility. But if probability is only an interpretation (merely a numerical expedient), then so must be its whole schema, including its backward computational procedure and the states of the world that it freezes, *including even the fact that there is such a thing as the maturity of the option, which is fixed in the future and from which we recede in the computational tree.*

The market models are only concerned with the market and with the ocean of prices. The market is essentially local, and there is no maturity in the market. As long as a derivative is trading, its only payoff is its next market price. Nobody needs to wait until maturity to get their money returned to them; all they have to do is unwind their position in the market at the next price. The exotic option price is a function of the prices of the hedging instruments at the present instant, and so it will be at the next. This describes a forward movement. It is only our representation of the way the market comes up with its prices that makes us propagate the procedure backwards, because the only link between now and next in our representation of the market is inter-temporal arbitrage and the stochastic processes we assume for the underliers. But things, events, can happen between now and next which break the current stochastic model and hence the constraints of inter-temporal non-arbitrage. This is called recalibration of the model, which we are very tempted, of course, to model probabilistically inside a meta-model (as a matter of fact, we have no other choice), and this is how we recuperate non-arbitrage and the backward valuation procedure back again. As a result, the pricing function of the market turns into a valuation algorithm back again. The pricing function is confused again with the mathematical function that represents its outputs. But what if the pricing function was a continually superior notion? Things happen between now and next which we do not and cannot model probabilistically; we recognize that the mathematical expression or representation of the pricing function (or its projection on the algorithmic, computational plane) is no longer the same; in other words, we recognize that the probabilistic model is no longer the same. The prices of the vanilla hedging instruments are all different, the price of the exotic option is of course different, and the mathematical expression now expressing the latter as a function of the former is also different. However, the pricing function *of the market* is still formally the same (despite its materialization as a specific mathematical expression having possibly varied), *because of our reiterated and reaffirmed trading decision.*

The market models are only concerned with the market and with the ocean of prices

The derivative value is a function of its underlying asset price at its expiration (either maturity, or knock-out), because its payoff is expressed as a mathematical function. But trading and the market are precisely what take place before the expiration. In the absence of the market and of a proper philosophical understanding (or overhaul) of the market, probability and stochastic processes of the underlying asset price are all we have. All they do is reproduce the valuation procedure and algorithm of the payoff function *before the expiry*, with the only difference that relations of intrinsic arbitrage (or static replication) that are valid at expiry are replaced by relations of non-intrinsic arbitrage (or dynamic replication). The existence of the market and the logic of the market models go precisely against that. The exotic option price is not a function of the prices of the vega-hedging instruments in the same way (same logic, same 'time') as it would be of the price of the underlying asset in the valuation and probabilistic logic. It is not derivative on the vega-hedging instruments. It depends on them because of present *trading* forces, relations, and therefore decisions. The prices of the vanilla options or forward variance contracts which we have selected as the vega-hedging instruments underlying our market model (i.e., whose price configuration we have decided would represent *the market*) influence and even cause the price of the exotic instrument to be what it is because of the market, not because of a programmed payoff schedule. At stake is nothing less than understanding what a derivatives market may be. Will it be condemned forever to be a schedule, a payoff valuation function, the output of an algorithm, or will it acquire a life and a force of its own? We have to make it, because derivative pricing is complex (it is mathematical – we cannot help it) and therefore requires market-makers; but how can we make it without necessarily reducing it to an algorithm? Is there a middle way? Market

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models achieve at least the first stage of the liberation, which is that the prices of the vega-hedging instruments are no longer equal to computed results but are given by the market. The task of the market-maker is now to produce the exotic price. It will be produced by a function; however, since its underliers are the vega-hedging instruments and the trading decision, this function is no longer an algorithm. It is a representation of the pricing function *of the market*, of what the market price of the exotic option would be, given the present market conditions and configuration (represented by the vanilla options or the forward variance contracts). We are not producing the exotic market price as an outcome of our non-arbitrage logic, but of the market's, because our pricing function is only temporarily, and not really, linked to stochastic structure. However, we are numerically making the market (i.e., responding to the numerical challenge due to the complexity of the derivatives), and the vanilla options or the forward variance contracts themselves are, by the way, eventually being recuperated as outputs of the same arbitrage-free model.

This way of making without making is definitional of the derivatives market, and affords the solution to its founding paradox by way of affirming it, not removing it. To repeat, there is no valuation theory anymore. The only theory is a pricing function. The valuation theory, which we still need in order to produce the theoretical arbitrage-free values (i.e., the numbers our equation or algorithm will be outputting), has been reduced and reinterpreted as an internal episode, which can vary, and will vary. Hence Bergomi's continued insistence on not hedging against the model-specific parameters. He speaks only of a 'correlation structure,' which the provisional (makeshift) model imposes on the prices of the hedging instruments, but which should not be taken more seriously than that. Precisely, the real pricing function (the market's) will never be written, because it is not a model; because it is real. Only models can be written, and they serve no other purpose than being internal episodes, or ways the pricing tool is temporarily made up. It is crucial that the ultimate pricing function should not be a model. For all we know, it could be the 'numerical twin' of multiple pricing models, which was trained in a neural network to recognize, from the surface of prices of the hedging instruments (the vanillas), the particular pricing model that has generated those prices and to recognize its parameters, and which reapplies this model to compute the exotic price. As this recognition was deep learned, we no longer see the underlying pricing model. All we see is the surface, the pricing function going from the surface of vanilla prices to the price of the exotic option. We can even extend the metaphor and claim that the *market's pricing function* is such a surface function without any recognizable underlying structure. In other words, it is literally that we now interpret Bergomi when he says that the stochastic representation we have selected for practical purposes is unimportant, and is expected to change.

The only thing we require from the pricing function is the decomposition of the P&L of the hedged position and the corresponding weakening of the modalities. The possibility of a break-even covariance matrix allows Bergomi to write the most general PDE and, as we said, it is only as an interpretation of this PDE that he writes stochastic processes for the prices of the hedging instruments (p. 219). The trading decision of Bergomi delays for ever (and the market is for ever) the moment when the exotic option becomes recognizably derivative on the underlying asset, and therefore has its price – in the backward pricing procedure – recognizably depend on state variables relative only to the underlying process, such as volatility and the volatility of volatility. We no longer go the maturity of the exotic option and come back; there is not even a notion of a backward procedure – the exotic option price depends on the present market prices of other instruments, and that's all. Before we reach maturity, countless changes of the price surface of the hedging instruments, as well as countless changes of the hypothetical stochastic structure (which is only an interpretation) supposed effectively to produce the practical computation of the exotic price through the practical representation of the unrepresentable pricing function, will have made it so that the only movement is forward.[11]

To be continued.

ENDNOTES

1. Bergomi, L. 2016. Stochastic Volatility Modeling. Boca Raton, FL: CRC Press, p. 72. All subsequent citations from Bergomi will be identified in the text by their page number in this edition.
2. "Let us now utilize the fact that our 'black box' valuation function *P* is in fact the local volatility price" (p. 68).
3. Bergomi, L. (2017). 'Local-stochastic volatility: models and non-models,' *Risk*, Aug, 78–83.
4. This is the decision to make the exotic option price a function of other derivatives' prices, instead of the result of an expectation calculation. See Bergomi (p. 16) as well as "God's Model vs. Market Models" (part I), *Wilmott* Sept, 34–47.
5. It is the solution of a PDE; however, probability (i.e., non-arbitrage) is an interpretation; for it is not sure the structure will not suddenly change by recalibration, thus breaking non-arbitrage. Bergomi's comment about discrete readjustments of P&L by parameters change is crucial, although seemingly parenthetical (p. 470).
6. See Ayache, E., Henrotte, P., Nassar, S., and Wang, X. (2004). 'Can anybody solve the smile problem?,' *Wilmott*, Jan, 78–96.
7. We know that the rough volatility proponents already have the answer to that. This means they are confident their 'true model' (Gatheral) prices correctly all kinds of exotics, and of exotics written on exotics, etc.
8. See "God's Model vs. Market Models" (part I), *Wilmott* Sept, 34–47.
9. In this, Bergomi weakens the modalities. Instead of requiring that the hedged position break even necessarily, he only requires that it shouldn't be impossible that it does. See "God's Model vs. Market Models" (part I) in endnote 8 above.
10. Ayache, E. (2015). The Medium of Contingency: An Inverse View of the Market. New York: Palgrave Macmillan, pp. 251–266.
11. Jeremy Bernstein finds odd that the pricing logic in BSM should flow backwards. He writes: "So why would I, as a physicist, find the Black–Scholes model quite odd? All physical theories are models. […] The object of [these models] is to predict the future. […] But the Black–Scholes model is quite different. It uses a model of the future to describe the present" (Bernstein, J. [2008]. Physicists on Wall Street and Other Essays on Science and Society. New York: Springer, p. 13).